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Non-Fourier Heat Conduction Modeling in a Finite Medium[1]

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A novel, simple iterative algorithm is used to calculate the temperature distribution in a finite medium for the case of non-Fourier (hyperbolic) heat conduction. In this algorithm the temperature is calculated explicitly in one simple calculation that is repeated for each time step as the heat wave propagates through the medium with constant speed. When the wave reaches a boundary of the medium, it bounces back and moves in the opposite direction. All simple initial and boundary conditions can be modelled. An example of using the algorithm for the case of a finite, thermally insulated medium is given, and the results are compared with an exact analytical solution.

KEY WORDS: non-Fourier heat conduction; numerical algorithm; temperature distribution.

1. INTRODUCTION

Heat flow ($\vec{q}$) in solids is generally regarded as a diffusion-like process, which is described by the Fourier law

$$\vec{q} = -\lambda \operatorname{grad} T, \tag{1}$$

where λ is the thermal conductivity and $\operatorname{grad} T$ is the temperature gradient. The temperature distribution is a solution of the classical (Fourier) heat conduction equation [1]:

$$\frac{\partial T}{\partial t} = \alpha \, \Delta T, \tag{2}$$

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where α is the thermal diffusivity, Δ is the Laplace operator and $T = T(\vec{r}, t)$ is the temperature at a space–time point $(\vec{r}, t)$. Equation (2) represents a partial differential equation (PDE) of the *parabolic* type, and its analytical solutions show a paradoxical behavior of infinite speed of propagation of the thermal disturbance. Any local change in temperature causes an instantaneous perturbation at each point of the medium, at whatever distance from the origin. It is in contradiction with the theory of relativity and also with known mechanisms of heat conduction.

Experiments with *second sound* in solid helium and in other crystalline solids [2], at very low temperatures [3], and at very short duration [4] clearly showed that the heat flows as a damped wave. If the crystal structure is almost defect-free (perfect) and the conditions for the second sound are met, then after pulse heating, an observable hump-like formation of increased temperature (*damped heat wave*) moves with a constant speed through the medium. The wave bounces back and forth from the boundaries while slowly dissipating its energy along the path. The damped heat wave is described by a PDE of the *hyperbolic* type first derived by Maxwell [5] and later postulated by Vernotte [6] and Cattaneo [7]

$$\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \alpha \, \Delta T, \tag{3}$$

where τ is the thermal relaxation time. The speed of propagation of the thermal wave is $v = \sqrt{\alpha/\tau}$. Equation (2) is a limiting case of Eq. (3) for $\tau \to 0$.

Analytical solutions for the non-Fourier Eq. (3) have been found only for a limited set of geometries and boundary conditions [8]. Existing standard software for numerical calculations of the temperature distribution is based mainly on the Fourier heat conduction equation. Therefore, it is hard to model non-Fourier heat conduction processes in real situations for engineers and designers without programming capabilities.

Recently, we have developed a novel, simple iterative algorithm for approximate calculation of the temperature distribution in a finite medium for Fourier heat conduction [9]. This paper will show that this algorithm can also be used to calculate the temperature distribution in a one-dimensional finite medium for non-Fourier heat transfer.

2. ANALYTICAL SOLUTION

The analytical solution of Eq. (3) for the temperature distribution in an isotropic homogeneous finite medium ($0 \leqslant x \leqslant L$), with zero initial temperature, adiabatically insulated boundaries, with one surface heated by a stepwise heat pulse of duration t_1 is given by [10]

$$V(x,t) = \frac{1}{t_1}\int_0^{t_1}\left[F(x,t-t') - \tau\frac{\partial F(x,t-t')}{\partial t'}\right]dt', \tag{4}$$

where

$$\begin{aligned} F(x,t) = {} & \frac{L}{\sqrt{\alpha\tau}}\exp\left(-\frac{t}{2\tau}\right)\sum_{k=0}^{\infty}\left\{I_0\left[\frac{1}{2\tau}\sqrt{t^2-\left(2kL+x\right)^2\frac{\tau}{\alpha}}\right]\right. \\ & \times H\left[t-(2kL+x)\sqrt{\frac{\tau}{\alpha}}\right] + I_0\left[\frac{1}{2\tau}\sqrt{t^2-\left(2kL+2L-x\right)^2\frac{\tau}{\alpha}}\right] \\ & \left.\times H\left[t-(2kL+2L-x)\sqrt{\frac{\tau}{\alpha}}\right]\right\}, \end{aligned} \tag{5}$$

$H(t)$ is a Heaviside unit step function, and $I_0(z)$ is a modified Bessel function of the first kind of zero order.

The analytical solution for a finite medium ($L = 1\,\text{cm}$, $\alpha = 0.025\,\text{cm}^2 \cdot \text{s}^{-1}$, $\tau = 10\,\text{s}$) heated by a stepwise pulse of duration $t_1 = 2\,\text{s}$, is shown in Fig. 1 Temperature profiles are calculated using Eq. (4) at three different times $t = t_1, 8t_1, 17t_1$. A dominant feature of this type of heat transfer is a thermal wave, which is travelling through the medium, bouncing back and forth from the boundaries, decaying exponentially with time, and dissipating its energy along its path.

3. ALGORITHM DESCRIPTION

In our algorithm for calculation of the temperature distribution, the medium is divided into N equal slabs of thickness $\Delta l = L/N$. These slabs are replaced by a perfect conductor of the same heat capacity separated by thermal resistance $\Delta l/\lambda$, so the temperature within a slab at any given time is constant. Heat propagates from one slab to another due to the existence of a temperature difference between the slabs. The wave takes a certain portion (given by the *inner transfer coefficient* $0 < \xi < 1$) of the excessive heat energy from one slab and moves that amount to the next one (redistribution), thus lowering the temperature difference between the two neighboring slabs. The wave starts from the left boundary slabs and proceeds in space from one pair of slabs to another, redistributing the thermal energy between the slabs. When it reaches the boundary of the medium, the wave bounces back and moves in the opposite direction in a perpetual manner.

Slab temperatures are $T_{i,m} \equiv T(x_i, t_m)$, where x_i $(i = 0, 1, 2, \ldots, N-1)$ is a spatial point (middle of the ith slab), and $t_m = m\,\Delta t$ $(m = 0, 1, 2, \ldots)$ is

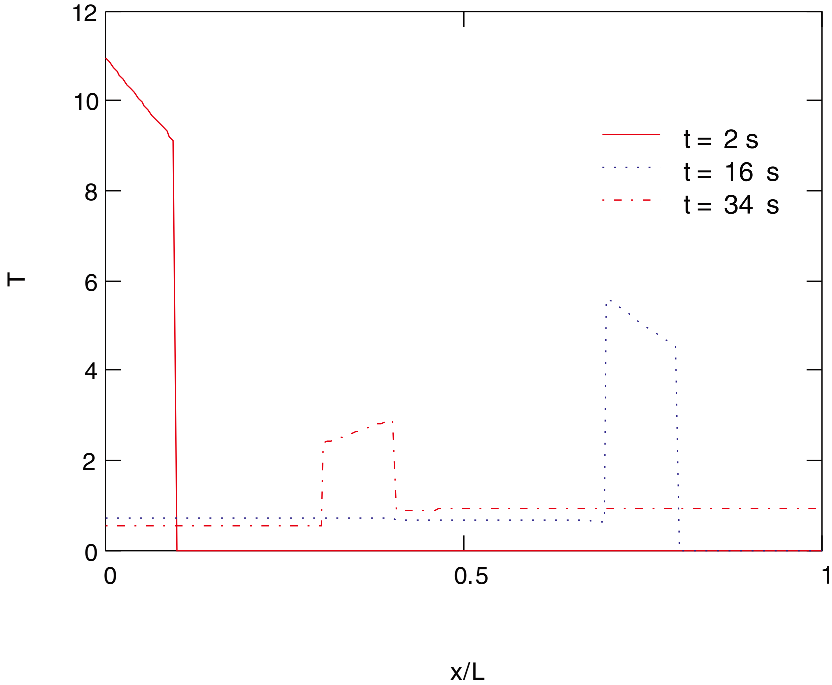

Fig. 1. Temperature distributions in a finite medium calculated using the analytical solution given by Eq. (4).

a discrete time point. The temperature distribution at time t_{m+1} when the heat wave is marching from left to right is given by

$$\begin{aligned} T_{n,m+1} &= T_{n,m} - \xi(T_{n,m} - T_{n+1,m})\delta_{n,m}, \\ T_{n+1,m+1} &= T_{n+1,m} + \xi(T_{n,m} - T_{n+1,m})\delta_{n,m}, \\ &\text{for } n = 0, 1, 2, \ldots, N-1, \end{aligned} \tag{6}$$

where $\delta_{n,m}$ is the Kronecker delta. The temperature of each slab changes twice as the wave passes the slab.

Similarly, the temperature distribution at time t_{m+1} when the heat wave is moving in the opposite direction from right to left is

$$\begin{aligned} T_{2N-n,m+1} &= T_{2N-n,m} - \xi(T_{2N-n,m} - T_{2N-n-1,m})\delta_{n,m}, \\ T_{2N-n-1,m+1} &= T_{2N-n-1,m} + \xi(T_{2N-n,m} - T_{2N-n-1,m})\delta_{n,m}, \\ &\text{for } n = N, N+1, N+2, \ldots, 2N-1. \end{aligned} \tag{7}$$

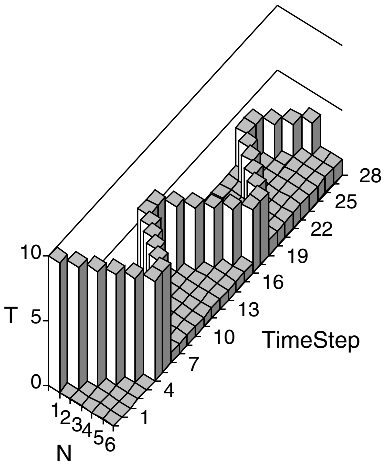

Fig. 2. Damped heat wave in a finite medium. Inner transfer coefficient $\xi = 0.95$.

An example of the heat wave in a one-dimensional finite medium with $\xi = 0.95$ is shown in Fig. 2. The medium is divided into $N = 6$ slabs. The wave height is reduced from 10 to 3.61 units after 28 time steps. The rest of the media is at about 1.2 units. The wave height decays exponentially with time, similarly as in the solution given by Eq. (4). The sum of all heights in the medium is always equal to 10, in accordance with the total energy conservation law.

When the heat wave imitates diffusion (parabolic heat transfer), then the wave is strongly damped ($\xi < 0.5$) and its actual position is not important. The time step Δt is therefore chosen to be equal to one loop time interval. On the contrary, the wave position is essential in the case of hyperbolic heat transfer, when the heat wave is much less damped ($\xi \to 1$) and moves across the medium with a constant speed. The time step in non-Fourier heat transfer has to be equal to the heat pulse duration ($\Delta t = t_1$). The time origin is also set to t_1 when the heat pulse already entered the medium. The slab thickness Δl is then given by

$$\Delta l = v t_1 = \sqrt{\frac{\alpha}{\tau}} t_1. \qquad (8)$$

Fraction $L/\Delta l$ defines the number of slabs N which should be an integer number, equal to or larger than five. In other words, the heat pulse duration t_1 should be at least five times less than the time $t^* = L\sqrt{\tau/\alpha}$, needed for the heat wave to reach the opposite end of the medium. These conditions limit the use of our algorithm, especially for long pulses, or very thin layers.

The inner transfer coefficient ξ for hyperbolic heat transfer is defined as

$$\xi = \left(1 + \frac{\Delta t}{2\tau}\right)^{-1}. \tag{9}$$

When the wave imitates non-Fourier heat transfer, the inner transfer coefficient is $1 > \xi \gtrsim 0.9$. It follows from Eq. 9) that the upper limit for the time step Δt is given by

$$\Delta t \lesssim \frac{2}{9}\tau. \tag{10}$$

This introduces a limit to the pulse duration in comparison with the relaxation time that can be modelled by our algorithm.

Figure 3a–c shows the temperature distributions in a finite medium for the case with $L = 1\,\text{cm}$, $\alpha = 0.025\,\text{cm}^2 \cdot \text{s}^{-1}$, $\tau = 10\,\text{s}$, and $t_1 = 2\,\text{s}$. The inner transfer coefficient is $\xi = 0.90909091$. The initial temperature (at $t = 2$ s) is 10 units for the first slab (from left), and the rest of the medium is at zero temperature. There are no heat losses at the boundaries. The temperature distribution calculated using our algorithm is compared with the exact analytical solution given by Eq. (4). In non-Fourier heat transfer it is important to know the temperature distribution in early stages after the heat pulse, so the profiles are calculated for (a) $t = 4\,\text{s}$, (b) $t = 18\,\text{s}$, and (c) $t = 32\,\text{s}$. It can be seen that the calculated temperatures are in good agreement with the analytical values.

If there are heat losses from the medium surfaces, a part of the excessive thermal energy leaves the medium, when the wave reaches the boundary slabs. Temperatures of the boundary slabs are furthermore changed due to the heat losses:

$$T_{N-1,N} = T_{N-1,N-1} - \zeta(T_{N-1,N-1} - T_A), \tag{11}$$

and

$$T_{0,2N+1} = T_{0,2N} - \zeta(T_{0,2N} - T_A), \tag{12}$$

where ζ is the *surface transfer coefficient* and T_A is the ambient temperature.

Various boundary conditions can be modelled by adjusting the surface transfer coefficient. In the case $\zeta = 1$ the constant temperature (equal

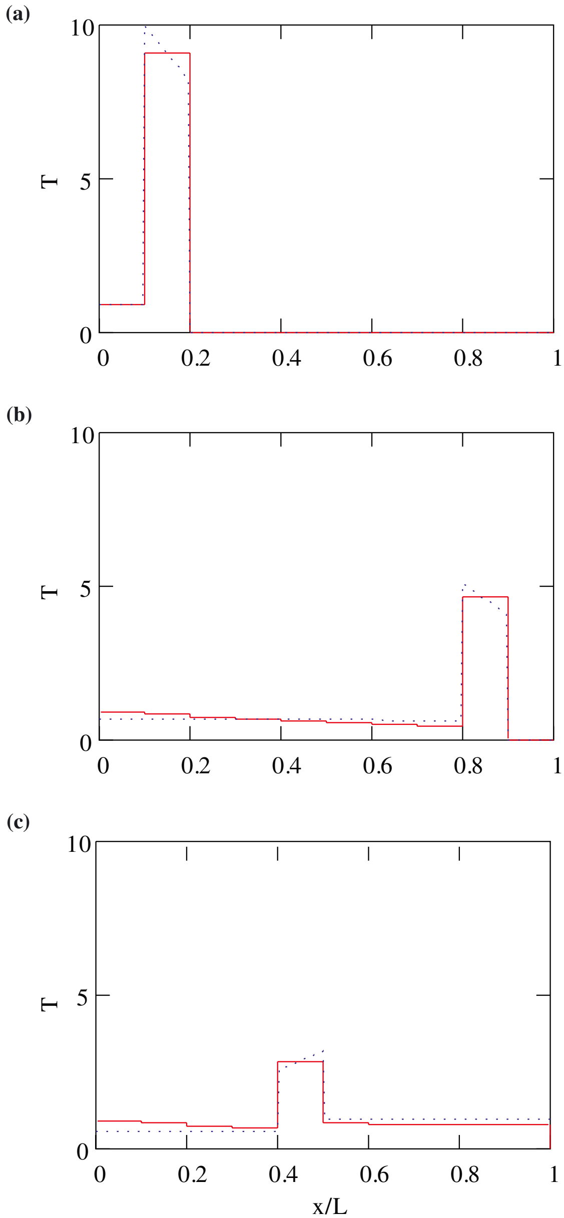

Fig. 3. Temperature distributions in a finite medium. Temperatures calculated using our algorithm (solid lines) are compared with exact analytical solutions (dotted lines) given by Eq. (4). The profiles are (a) at $t = 4\,\text{s}$; (b) at $t = 18\,\text{s}$; and (c) at $t = 32\,\text{s}$.

to the ambient temperature) boundary condition is being simulated. An adiabatically insulated surface is given by $\zeta = 0$. Non-linear conditions can be also modelled, e.g., radiation from the surface, in which the rate of heat energy leaving the surface is proportional to

$$\sigma\epsilon(T_i^4 - T_{\mathrm{A}}^4),$$

where σ is the Stefan–Boltzmann constant and ϵ is the emissivity of the surface. The temperature difference of the fourth power of the temperatures will be used in Eqs. (11) and (12).

4. CONCLUSION

The temperature distribution in a finite medium for the case of non-Fourier heat conduction can be calculated using a simple iterative algorithm. In this algorithm the temperature is calculated explicitly in one simple calculation that is repeated for each time step as the heat wave proceeds through the medium with a constant speed.

The proposed algorithm can be used by engineers and designers as a fast, easy to understand, and easy to implement alternative to existing numerical and analytical methods. It could simplify hardware and software needs for temperature and heat flux calculations in real applications and open new possibilities for improving measurement and nondestructive testing procedures used in this field.

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